

A Systematic Framework for Composite Hypothesis Testing of Independent Bernoulli Trials

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Abstract—This letter is focused on the classic problem of testing samples drawn from independent Bernoulli probability mass functions, when the success probability under the alternative hypothesis is not known. The goal is to provide a systematic taxonomy of the viable detectors (designed according to theoretically-founded criteria) which can be used for the specific instance of the problem. Both One-Sided (OS) and Two-Sided (TS) tests are considered, with reference to: (i) identical success probability (a homogeneous scenario) or (ii) different success probabilities (a non-homogeneous scenario) for the observed samples. As a result of the study, a complete summary (in tabular form) of the relevant statistics for the problem is provided, along with a discussion on the existence of the Uniformly Most Powerful (UMP) test. Finally, when the Likelihood Ratio Test (LRT) is not UMP, existence of the UMP detector after reduction by invariance is investigated.

Index Terms—Composite Hypothesis Testing, Decision Fusion, Binary Integration, Invariant Detection.

I. PROBLEM FORMULATION

HYPOTHESIS testing from Bernoulli samples is of extreme importance for signal processing. Relevant applications range from radar detection (for binary integration [1], [2]) to sensor fusion [3], [4], from reliability theory [5] to pattern identification [6].

More specifically, the problem can be formulated as a binary hypothesis test, where K observations¹ y_k (collected in the vector \mathbf{y}) are used to discriminate between the null (\mathcal{H}_0) and the alternative (\mathcal{H}_1) hypotheses. The k th observation, $k \in \mathcal{K} \triangleq \{1, 2, \dots, K\}$, is characterized by the conditional probabilities $P(y_k|\mathcal{H}_j)$. For notational convenience we denote $P_{1,k} \triangleq P(y_k = 1|\mathcal{H}_1) \in (0, 1)$ and $P_{0,k} \triangleq P(y_k = 1|\mathcal{H}_0) \in$

$(0, 1)$ the success probabilities under \mathcal{H}_1 and \mathcal{H}_0 , respectively, and we also assume conditionally independent observations, i.e. $P(\mathbf{y}|\mathcal{H}_j) = \prod_{k=1}^K P(y_k|\mathcal{H}_j)$.

As for $P_{0,k}$, they can be either *known or unknown*, whereas $P_{1,k}$ are *unknown* and deterministic parameters. The reason is that the alternative hypothesis \mathcal{H}_1 commonly models events such as the target presence in radar detection or the occurring of a specific phenomenon in Sensor Networks (SNs); hence the mentioned probabilities usually depend on some unknown model parameters [2], [4], [8], [9].

The measurement model is entirely specified via the joint multivariate probability mass function (pmf) under the generic hypothesis \mathcal{H}_i , denoted as $P(\mathbf{y}; P_{i,1}, \dots, P_{i,K})$. In many relevant cases, $P_{1,k}$ and $P_{0,k}$ have the meaning of *detection* and *false alarm* probabilities, thus it can be safely assumed that $P_{1,k} \geq P_{0,k}$, since each meaningful detector has performance above the so-called “chance line” [2], [10]. On the other hand, in contexts such as that of [9], [11], $P_{1,k}$ is the result of a one-bit (dumb) quantization of a raw measurement under \mathcal{H}_1 . More specifically, in the latter case the model $y_k = \text{sign}(h_k \theta + w_k - \tau_k)$ holds for \mathcal{H}_1 , where h_k , θ , w_k and τ_k are an observation coefficient, a real-valued unknown parameter, a noise term with zero-mean and unimodal probability density function and a suitable threshold, respectively. Then, it is of interest considering both One-Sided (OS) and Two-Sided (TS) testing [10]. Summarizing, the considered problems are:

$$\text{(OS)} \quad \mathcal{H}_0 : P_k = P_{0,k}; \quad \mathcal{H}_1 : P_k > P_{0,k}; \quad (1)$$

$$\text{(TS)} \quad \mathcal{H}_0 : P_k = P_{0,k}; \quad \mathcal{H}_1 : P_k \neq P_{0,k}; \quad (2)$$

which will be studied with reference to the following two scenarios: (i) *homogeneous* with $(P_{1,k}, P_{0,k}) = (P_1, P_0)$, $k \in \mathcal{K}$; (ii) *non-homogeneous* with arbitrary $(P_{1,k}, P_{0,k})$, $k \in \mathcal{K}$ (as a byproduct we will also analyze the *partially-homogeneous* scenario, that is $P_{0,k} = P_0$, $k \in \mathcal{K}$).

The aim of this letter is to provide a detailed overview of the possible alternatives which can be tackled for the composite hypothesis testing specified in Eqs. (1) and (2), especially when the optimum Log-Likelihood Ratio (LLR) cannot be implemented. More specifically, we first investigate the existence of the Uniformly Most Powerful (UMP) test pointing out that *it does not exist*, with a few exceptions. Consequently, we derive the Generalized Likelihood Ratio Test (GLRT), the Rao Test, the Wald Test and the Locally Most (Mean) Powerful Test (LM(M)PT) as viable decision strategies (also investigating and discussing possible coincidence and/or statistical equivalence). Furthermore, we also focus on the existence of a UMP Invariant (UMPI) test under the group of transformation composed of the permutations of the samples.

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⁰Notation - Lower-case bold letters denote vectors, with a_n being the n th element of \mathbf{a} ; upper-case calligraphic letters, e.g. \mathcal{A} , denote finite sets; $\mathbb{E}\{\cdot\}$ and $(\cdot)^T$ denote expectation and transpose, respectively; $\text{diag}(a_1, \dots, a_K)$ denotes a $K \times K$ diagonal matrix with elements a_i on the main diagonal; $P(\cdot)$ denotes probability mass functions (pmf), while $P(\cdot|\cdot)$ is the corresponding conditional counterparts; $\mathcal{B}(p)$ denotes a Bernoulli pmf with success probability p ; $u(\cdot)$ denotes the Heaviside unit-step function; finally the symbol \sim means “distributed as” and “ \propto ” is used to underline statistical equivalence between decision statistics.

¹In this paper we will adopt the generic term “observation”. Indeed, depending on the specific context, this may have different meanings. For example, it could represent the binary decision of a sensor in decentralized detection [7]. Alternatively, this may be related to the non-coherent decision output of a single pulse within a train of pulses in radar detection via binary integration [2].

Remarkably, invariance arguments lead to some optimality claims for the well-known Counting Rule (CR) [2], [7], which for OS testing turns out to be the UMPI detector². This represents a solid justification for its application when only very limited knowledge about the hypothesis testing problem is available. Finally, we provide a systematic taxonomy for the considered detectors in all the investigated scenarios.

The remainder of this letter is organized as follows: in Sec. II we derive and discuss the main results, while in Sec. III we draw some concluding remarks.

II. TAXONOMY OF DECISION RULES

A. LLR and UMP test

We start from the analysis of the simple homogeneous scenario. According to Neyman-Pearson criterion, the LLR is derived easily as [10]:

$$\begin{aligned} \Lambda_{\text{LLR}} &\triangleq \ln \left[\frac{P(\mathbf{y}; P_1)}{P(\mathbf{y}; P_0)} \right] = \sum_{k=1}^K \ln \left[\frac{P(y_k; P_1)}{P(y_k; P_0)} \right] \\ &= \sum_{k=1}^K \left\{ y_k \ln \left[\frac{P_1}{P_0} \right] + (1 - y_k) \ln \left[\frac{1 - P_1}{1 - P_0} \right] \right\} \end{aligned} \quad (3)$$

and can be shown to be statistically equivalent to:

$$\Lambda_{\text{LLR}} \propto \ln \left[\frac{P_1(1 - P_0)}{P_0(1 - P_1)} \right] \sum_{k=1}^K y_k \quad (4)$$

Eq. (4) evidently depends on the unknown parameter P_1 . Nonetheless, a careful inspection reveals that in OS testing (cf. Eq. (1)) the condition $P_1 > P_0$ automatically implies positivity of log term in Eq. (4). Therefore, the CR [3], [4] is UMP in OS testing, as a consequence of Karlin-Rubin theorem [12]. On the other hand, it is easily understood that the *UMP test does not exist* when testing a TS alternative.

Similarly, in the non-homogeneous case, the LLR is obtained as follows:

$$\Lambda_{\text{LLR}} = \sum_{k=1}^K \left\{ y_k \ln \left[\frac{P_{1,k}}{P_{0,k}} \right] + (1 - y_k) \ln \left[\frac{1 - P_{1,k}}{1 - P_{0,k}} \right] \right\} \quad (5)$$

Such a rule is very common in statistical literature and, in decision fusion applications, it is commonly referred to as the Chair-Varshney rule [7]. Clearly, the LLR cannot be evaluated when the $P_{1,k}$'s are unknown. Also, it is apparent that for both TS and OS testing the *UMP test does not exist*. For such a reason, evaluation and comparison of viable decision rules is relevant and will be the object of the remainder of the manuscript.

²In this specific case, if a partially-homogeneous scenario is considered, statistical equivalence among the UMPI, GLR, Rao and LMMP detectors also arises.

B. GLRT

The GLRT is widely used to devise decision rules in composite hypothesis testing [10]. In the present setup the GLR for the homogeneous case is given by

$$\Lambda_G \triangleq \ln \left[\frac{\max_{\rho} P(\mathbf{y}; \rho)}{P(\mathbf{y}; P_0)} \right], \quad (6)$$

and assumes the explicit expression:

$$\Lambda_G = \begin{cases} K \mathcal{D}_{\text{KL}}(\mathcal{B}(\hat{P}) || \mathcal{B}(P_0)) u(\hat{P} - P_0) & \text{(OS)} \\ K \mathcal{D}_{\text{KL}}(\mathcal{B}(\hat{P}) || \mathcal{B}(P_0)) & \text{(TS)} \end{cases}; \quad (7)$$

where $\mathcal{D}_{\text{KL}}(\cdot || \cdot)$ denotes the Kullback-Leibler (KL) divergence [13] and $\hat{P} \triangleq \frac{\sum_{k=1}^K y_k}{K}$. The well-known test proposed by Hoeffding in [14] arises in TS testing, i.e. a threshold test based on the sample KL divergence between the Bernoulli pmfs in the two hypotheses. Similarly, in OS testing a restricted version for $\hat{P} > P_0$ is obtained.

Differently, in the non-homogeneous scenario, GLR requires the following parameter-vector maximization:

$$\Lambda_G \triangleq \ln \left[\frac{\max_{\rho} P(\mathbf{y}; \rho)}{P(\mathbf{y}; \rho_0)} \right] \quad (8)$$

where we have denoted $\rho \triangleq [\rho_1 \ \cdots \ \rho_K]^T$ and $\rho_0 \triangleq [P_{0,1} \ \cdots \ P_{0,K}]^T$, respectively. It is not difficult to show that k th element of the ML estimate $\hat{\rho}$ (under \mathcal{H}_1) is:

$$\hat{\rho}_k \triangleq \begin{cases} \max\{P_{0,k}, y_k\} & \text{(OS)} \\ y_k & \text{(TS)} \end{cases}. \quad (9)$$

Substituting Eq. (9) into (8) leads to $\Lambda_G = \sum_{k=1}^K \lambda_{G,k}$, where³:

$$\lambda_{G,k} \triangleq \begin{cases} y_k \ln \left[\frac{1}{P_{0,k}} \right] & \text{(OS)} \\ y_k \ln \left[\frac{1 - P_{0,k}}{P_{0,k}} \right] & \text{(TS)} \end{cases} \quad (10)$$

Finally, it is worth noticing that in the partially-homogeneous scenario ($P_{0,k} = P_0$), the *GLR coincides with the CR* in OS testing, while the (very mild) condition $P_0 < \frac{1}{2}$ is needed in TS testing.

C. Rao Test

In the homogeneous case, the Rao (score) test is evaluated as [9], [15]:

$$\Lambda_R \triangleq \left. \frac{(\partial \ln [P(\mathbf{y}; \rho)] / \partial \rho)^2}{\mathcal{I}(\rho)} \right|_{\rho=P_0} \quad (11)$$

$$= \frac{K(\hat{P} - P_0)^2}{P_0(1 - P_0)} \quad (12)$$

$$= \frac{K}{P_0(1 - P_0)} \mathcal{D}_{\text{TVD}}(\mathcal{B}(\hat{P}) || \mathcal{B}(P_0))^2 \quad (13)$$

where $\mathcal{I}(\rho) \triangleq \mathbb{E} \left\{ (\partial \ln [P(\mathbf{y}; \rho)] / \partial \rho)^2 \right\}$ is the Fisher information and $\mathcal{D}_{\text{TVD}}(\cdot)$ denotes the Total Variation Distance

³We have exploited statistical equivalence in order to neglect the irrelevant term $\sum_{k=1}^K \ln \left(\frac{1}{1 - P_{0,k}} \right)$ in TS testing.

Table I
 TABLE FOR DETECTORS COMPARISON: (H) HOMOGENEOUS SCENARIO (P_1, P_0); (NH) NON-HOMOGENEOUS SCENARIO ($P_{0,k}, P_{1,k}$); (PH)
 PARTIALLY-HOMOGENEOUS SCENARIO ($P_{0,k}, P_1$). (*) $\sum_{k=1}^K y_k$ IF $P_0 < \frac{1}{2}$.

	<i>UMP Test</i>	<i>UMPI Test</i>	<i>GLRT</i>	<i>Rao Test</i>	<i>L(M)MPT</i>	<i>Wald Test</i>
OS (H)	$\sum_{k=1}^K y_k$	$\sum_{k=1}^K y_k$	$K \mathcal{D}_{\text{KL}}(\mathcal{B}(\hat{P}) \mathcal{B}(P_0)) u(\hat{P} - P_0)$	$(\hat{P} - P_0)^2$	$\sum_{k=1}^K y_k$	$\frac{K(\hat{P}_+ - P_0)^2}{\hat{P}_+(1 - \hat{P}_+)}$
TS (H)	X	X	$K \mathcal{D}_{\text{KL}}(\mathcal{B}(\hat{P}) \mathcal{B}(P_0))$	$(\hat{P} - P_0)^2$	X	$\frac{K(\hat{P} - P_0)^2}{\hat{P}(1 - \hat{P})}$
OS (PH)	X	$\sum_{k=1}^K y_k$	$\sum_{k=1}^K y_k$	$\sum_{k=1}^K \frac{(y_k - P_0)^2}{P_0(1 - P_0)}$ (*)	$\sum_{k=1}^K y_k$	X
TS (PH)	X	X	$\sum_{k=1}^K y_k$ if $P_0 < \frac{1}{2}$	$\sum_{k=1}^K \frac{(y_k - P_0)^2}{P_0(1 - P_0)}$ (*)	X	X
OS (NH)	X	$\sum_{k=1}^K y_k$	$\sum_{k=1}^K y_k \ln\left(\frac{1}{P_{0,k}}\right)$	$\sum_{k=1}^K \frac{(y_k - P_{0,k})^2}{P_{0,k}(1 - P_{0,k})}$	$\frac{\sum_{k=1}^K (y_k - P_{0,k})}{\sqrt{\sum_{k=1}^K P_{0,k}(1 - P_{0,k})}}$	X
TS (NH)	X	X	$\sum_{k=1}^K y_k \ln\left(\frac{1 - P_{0,k}}{P_{0,k}}\right)$	$\sum_{k=1}^K \frac{(y_k - P_{0,k})^2}{P_{0,k}(1 - P_{0,k})}$	X	X

(TVD). We notice that statistical equivalence of Eq. (12) to the CR *cannot be claimed* since

$$\frac{K(\hat{P} - P_0)^2}{P_0(1 - P_0)} \propto (\hat{P} - P_0)^2 \propto \hat{P}(\hat{P} - 2P_0). \quad (14)$$

Similarly, the Rao test in the non-homogeneous case is evaluated as:

$$\Lambda_R \triangleq \left\{ \frac{\partial \ln [P(\mathbf{y}; \boldsymbol{\rho})]^T}{\partial \boldsymbol{\rho}} \mathcal{I}(\boldsymbol{\rho})^{-1} \frac{\partial \ln [P(\mathbf{y}; \boldsymbol{\rho})]}{\partial \boldsymbol{\rho}} \right\} \Bigg|_{\boldsymbol{\rho}=\boldsymbol{\rho}_0} \quad (15)$$

$$= \sum_{k=1}^K \frac{(y_k - P_{0,k})^2}{P_{0,k}(1 - P_{0,k})} \quad (16)$$

Finally, with reference to the partially-homogeneous scenario ($P_{0,k} = P_0$), we observe from Eq. (16) that Λ_R is statistically equivalent to the CR in TS testing if $P_0 < \frac{1}{2}$ (cf. Eq. (16)), since

$$\sum_{k=1}^K \frac{(y_k - P_0)^2}{P_0(1 - P_0)} \propto \sum_{k=1}^K (y_k - P_0)^2 \propto \sum_{k=1}^K y_k(1 - 2P_0). \quad (17)$$

D. Locally Most (Mean) Powerful Test (LM(M)PT)

Since we are also considering OS testing, a LMPT seems appropriate in this context. Indeed, for homogeneous scenario and OS testing, the LMPT is obtained as [10]

$$\Lambda_L \triangleq \frac{\partial \ln [P(\mathbf{y}; \boldsymbol{\rho})] / \partial \boldsymbol{\rho}}{\sqrt{\mathcal{I}(\boldsymbol{\rho})}} \Bigg|_{\boldsymbol{\rho}=P_0} \quad (18)$$

$$= \frac{\sqrt{K}(\hat{P} - P_0)}{\sqrt{P_0(1 - P_0)}} \propto \hat{P} \propto \sum_{k=1}^K y_k \quad (19)$$

where last expression underlines *statistical equivalence* (independently on the value of P_0) to the CR.

On the other hand, in the non-homogeneous scenario, it can be shown that a LMPT cannot be obtained [10]. Indeed, the first-order Taylor series of the LLR depends on the (unknown) differences ($P_{1,k} - P_{0,k}$), which weight the gradient (score) vector. Therefore, to overcome this issue, we resort to a modified multi-dimensional version of the LMPT, which

maximizes the mean curvature of the power function in the neighborhood of $\boldsymbol{\rho}_0$, that is [16], [17]

$$\Lambda_L \triangleq \frac{\sum_{k=1}^K (y_k - P_{0,k})}{\sqrt{\sum_{k=1}^K P_{0,k}(1 - P_{0,k})}} \quad (20)$$

Statistic in Eq. (20) is usually referred⁴ to be Locally Most Mean Powerful (LMMP). Finally, with reference to the partially-homogeneous scenario ($P_{0,k} = P_0$), we observe that similar considerations as in the homogeneous case apply, that is, Λ_L is statistically equivalent to the CR for OS testing (cf. Eq. (20)).

E. Wald Test

The well-known Wald Test in the homogeneous case is [15]

$$\Lambda_w \triangleq (\hat{P}_+ - P_0)^2 \mathcal{I}(\hat{P}_+) = \frac{K(\hat{P}_+ - P_0)^2}{\hat{P}_+(1 - \hat{P}_+)}, \quad (21)$$

where \hat{P}_+ is defined as:

$$\hat{P}_+ \triangleq \begin{cases} \max\{P_0, \hat{P}\} & \text{(OS)} \\ \hat{P} & \text{(TS)} \end{cases}. \quad (22)$$

On the other hand, we remark that in a non-homogeneous situation the Wald test *cannot be constructed*. Indeed, in the latter case the general expression is:

$$\Lambda_w \triangleq (\hat{\boldsymbol{\rho}} - \boldsymbol{\rho}_0)^T \mathcal{I}(\hat{\boldsymbol{\rho}}) (\hat{\boldsymbol{\rho}} - \boldsymbol{\rho}_0) \quad (23)$$

where $\hat{\boldsymbol{\rho}}$ has its elements defined in Eq. (9). Unfortunately, the i th component of the Fisher matrix $\mathcal{I}(\boldsymbol{\rho}) = K \text{diag}\left(\frac{1}{\rho_1(1-\rho_1)}, \dots, \frac{1}{\rho_K(1-\rho_K)}\right)$ diverges when the corresponding component of the ML estimate $\hat{\boldsymbol{\rho}}$ equals 0 or 1 in a TS testing (1 in a OS testing, respectively), thus making the statistic in Eq. (23) not applicable.

⁴With a slight abuse of notation we will use the symbol Λ_L to denote both LMPT and LMMP, depending on the specific scenario.

F. Reduction by Invariance and Existence of the UMPI Test

As already pointed out, the problem at hand admits a UMP test only in very special cases. Therefore it is convenient pursuing alternative approaches. To this end, we resort to the theory of invariance, which allows focusing on decision rules exhibiting some natural symmetries implying important practical properties [18]. Besides, the use of invariance leads to a data reduction because all invariant tests can be expressed in terms of statistics, called the Maximal Invariant Statistics (MIS), denoted hereinafter with $T(\mathbf{y})$, which organizes the original data into equivalence classes. Mathematically speaking, given the group of transformations \mathcal{G} , a MIS has the following properties: (i) $T(\mathbf{y}) = T(g(\mathbf{y}))$ for all $g \in \mathcal{G}$; (ii) if $T(\mathbf{y}_a) = T(\mathbf{y}_b)$, then $\mathbf{y}_a = g(\mathbf{y}_b)$ for some $g \in \mathcal{G}$.

First, as a meaningful choice of the invariance group, we choose to devise tests which are independent on the association between the sample y_k and the corresponding couple $(P_{1,k}, P_{0,k})$, that is, we define \mathcal{G} as the set of permutations of the vector \mathbf{y} . Then, it is not difficult to show that for the considered problem (in both homogeneous and non-homogeneous scenarios) the MIS is scalar and coincides with the CR. This property holds since $T(\mathbf{y}) = k$, for each vector containing k bits set to 1. Also, it can be always found a permutation transforming \mathbf{y}_a into \mathbf{y}_b , if $T(\mathbf{y}_a) = T(\mathbf{y}_b) = k$ (i.e. a simple bit “shuffling”).

Once we have proved that the CR is the MIS, we can build the Most Powerful Invariant (MPI) test, based on the decision statistic

$$\Lambda_{\text{MPI}} \triangleq \ln \left[\frac{P(T; \boldsymbol{\rho}_1)}{P(T; \boldsymbol{\rho}_0)} \right] \quad (24)$$

where $\boldsymbol{\rho}_1 \triangleq [P_{1,1} \ \cdots \ P_{1,K}]^T$, which is clearly the LLR of the MIS. It is not difficult to show that in the simple homogeneous scenario $P(T; \boldsymbol{\rho}_i)$ is a Binomial pmf, while in the non-homogeneous case $P(T; \boldsymbol{\rho}_i)$ generalizes to a Poisson-Binomial pmf [19], [20], [21], whose expression is given by⁵:

$$P(T = \ell | \mathcal{H}_i) = \sum_{\mathbf{y}: T(\mathbf{y}) = \ell} \prod_{k=1}^K (P_{i,k})^{y_k} \prod_{s=1}^K (1 - P_{i,s})^{(1-y_s)}. \quad (25)$$

Thus, assuming $\sum_{k=1}^K y_k = \ell$, the MPI is evaluated as:

$$\Lambda_{\text{MPI}} = \ln \left\{ \frac{\sum_{\mathbf{y}: T(\mathbf{y}) = \ell} \prod_{k=1}^K (P_{1,k})^{y_k} \prod_{s=1}^K (1 - P_{1,s})^{(1-y_s)}}{\sum_{\mathbf{y}: T(\mathbf{y}) = \ell} \prod_{k=1}^K (P_{0,k})^{y_k} \prod_{s=1}^K (1 - P_{0,s})^{(1-y_s)}} \right\} \quad (26)$$

Clearly, Eq. (26) cannot be implemented since $P_{1,k}$'s are not known. However, we can check for existence of UMPI test. To proceed further, we distinguish between the two situations:

- (OS testing): The *UMPI test does exist*. Indeed, it can be shown, under the assumption $P_{1,k} > P_{0,k}$ (the proof

is given in [22, Thm. 2]), that the MPI decision statistic Λ_{MPI} in Eq. (24) is an increasing function of $T(\mathbf{y})$. Then, by invoking Karlin-Rubin theorem, we can state that the *CR is the UMPI test*. The latter statement clearly holds for both homogeneous and non-homogeneous scenarios.

- (TS testing): the *UMPI test does not exist*. In fact, even in the simpler homogeneous scenario Λ_{MPI} is:

$$\begin{aligned} \Lambda_{\text{MPI}}(T = \ell) &= \ln \left[\frac{\binom{K}{\ell} P_1^\ell (1 - P_1)^{K-\ell}}{\binom{K}{\ell} P_0^\ell (1 - P_0)^{K-\ell}} \right] \\ &\propto \ell \ln \left[\frac{P_1 (1 - P_0)}{P_0 (1 - P_1)} \right] \end{aligned} \quad (27)$$

This implies that the MPI decision statistic is not always increasing with the MIS ℓ (namely the CR), since this depends on whether $P_1 \geq P_0$. Similar reasoning can be used to prove analogous statement in a non-homogeneous scenario.

G. Detectors Discussion

A complete summary of the viable decision statistics is reported in Tab. I. It is apparent that in OS testing (which is relevant for radar and decision fusion applications), CR should be used in a homogeneous scenario, since it is the UMP test. Differently, in non-homogeneous and partially-homogeneous cases, GLRT can be employed as a suitable test. Alternatively, Rao, Wald (in the homogeneous case) and L(M)MPT tests can be used; their relative performance will of course depend from case to case. However, CR has been shown to be: (i) UMPI in all the aforementioned scenarios; (ii) statistically equivalent to GLRT and Rao test (under the mild condition $P_0 < \frac{1}{2}$) in a partially-homogeneous scenario and (iii) statistically equivalent to L(M)MPT in a (partially) homogeneous scenario. Such results confirm its robustness observed in the open literature mainly via simulations.

On the other hand, for TS testing we have shown that neither the UMP nor the UMPI tests exist. As for the latter, the GLRT can be used, as well as Rao and Wald (in a homogeneous scenario) tests; same considerations as in OS testing apply on their performance. Finally, CR has been shown to be statistically equivalent to both GLRT and Rao test in a partially-homogeneous scenario (under the mild condition $P_0 < \frac{1}{2}$).

III. CONCLUSIONS

In this letter we provided an overview of the classic problem of testing samples drawn from independent homogeneous (non-homogeneous) Bernoulli pmfs, when the success probability (probabilities) under the alternative hypothesis is (are) not known. Both OS and TS testing were considered in our analysis. Existence of the UMP test was investigated and confirmed only in OS testing with a homogeneous scenario. Therefore, GLRT, Rao test, Wald test and LM(M)PT were derived and their possible coincidence and/or statistical equivalences to the well-known CR underlined. Moreover, the existence of the UMPI test was investigated under the group of transformations represented by the samples permutations. With reference to the invariance domain, it was shown that in OS testing the UMPI test is the CR itself.

⁵It is worth noticing that Eq. (25) requires sums which are infeasible to compute in practice unless K is small. For this reason different methods have been proposed in literature for its efficient evaluation. The alternatives include fast convolution of individual Bernoulli pmfs [19], recursive approaches [20] and a Discrete Fourier Transform based computation [21].

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