On MSE Performance of Time-Reversal MUSIC

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Abstract—In this paper we study the performance of time-reversal multiple signal classification (TR-MUSIC) for computational TR applications. The analysis builds upon classical results on first-order perturbation of singular value decomposition. The closed form of mean-squared error (MSE) matrix of TR-MUSIC is derived for a narrowband multistatic co-located scenario and is compared with both numerical simulations and the Cramér-Rao lower bound.

I. INTRODUCTION

The time reversal (TR) approach finds several applications such as medical imaging, non-destructive testing and ground penetrating radar. The key idea behind the so-called physical TR methods is to record a signal emitted by sources or reflected by targets using an array of transducers; the time-reversed and complex conjugated measurements are then transmitted back into the medium. In a reciprocal medium, the backpropagated wave will then retrace the original trajectory and focus around the original source locations without the need to solve the inverse of the channel. A popular application of TR “refocusing property” is locating targets by computational TR through imaging [1], [2], [3], [4]. In the latter case, after receiving the signal reflected from the target, a backpropagated process is computed rather than implemented in the real medium. TR multiple signal classification (TR-MUSIC), first introduced for the Born approximated scattering (BA) (linear) model, is one of the mentioned approaches [1]. The maximum-likelihood estimator (MLE) and other sub-optimal estimators for computational TR were presented in [2], both for BA and Foldy-Lax (FL) (non linear) models. The additional task of estimating scattering potential via a non-iterative (approximate) formula is addressed in [4] for location-only estimators. A theoretical study on performance, based on the Cramér-Rao lower bound (CRLB), was presented in [3], both for BA and FL models. Sensitivity analysis of several computational TR techniques to non-matching assumptions is studied in [5].

A vast literature on the performance analysis of MUSIC for direction-of-arrival (DOA) estimation is present [6]. Performance analysis in terms of resolution was pioneered by [7] for a simple scenario, while a detailed analysis of MUSIC mean-squared error (MSE) can be found in [8]. The MSE/bias analysis in presence of modeling errors has been considered with a second-order subspace perturbation in [9], while a MSE/bias analysis, conditioned on the resolution event, was recently introduced in [10]. Unluckily, to the best of our knowledge, no such corresponding theoretical results are present in the literature for TR-MUSIC.

Here we provide a theoretical performance analysis of TR-MUSIC in terms of the MSE matrix of the position estimates, via perturbation of singular value decomposition (SVD). A co-located multistatic (narrowband) scenario with either BA or FL scattering is considered in this paper. The presented results complement those found in DOA literature [6] and highlight: (i) performance dependence of TR-MUSIC on the scatterers configuration and (ii) its inherent limitations. It is shown that the CRLB, though being representative of MLE performance, does not accurately predict TR-MUSIC performance, as opposed to the proposed expression. The latter is further used to compare the asymptotic MSE attainable under both BA and FL models.

II. SYSTEM MODEL

The system model is described as follows. We consider point scatterers and a multistatic setup: \( M \) scatterers located at unknown positions \( \{x_k\}_{k=1}^M \in \mathbb{R}^P \) with unknown scattering potentials \( \{\tau_k\}_{k=1}^M \in \mathbb{C} \); a co-located transmit/receive (Tx/Rx) array with \( N \) isotropic point elements located at the points \( r_i \in \mathbb{R}^P \), \( i \in \{1, \ldots, N\} \). The illuminators first send signals to the probed scenario (in a known homogeneous background with wavenumber \( \kappa^2 \)) and the transducer array records the received signals. The measurement model is then [2]:

\[
K_k = K(x_{1:M}, \tau) + W
\]

\[
= G(x_{1:M}, \tau) G(x_{1:M})^H + W
\]

where \( K(x_{1:M}, \tau) \in \mathbb{C}^{N \times N} \) and \( K_k \in \mathbb{C}^{N \times N} \) denote the multistatic response matrix (MSR) in frequency-domain and the measurement matrix, respectively. Also \( W \in \mathbb{C}^{N \times N} \) is a noise matrix whose elements \( \text{vec}(W) \sim \mathcal{N}_{\mathbb{C}}(0, \sigma^2_{W} I_N^2) \). Finally, we have denoted the vector of scattering coefficients \( a \).

2Notation—Lower-case (resp. Upper-case) bold letters denote column vectors (resp. matrices), with \( a_n \) (resp. \( a_{n,m} \)) being the \( n \)-th (resp. the \( (n,m) \)-th) element of \( a \) (resp. \( A \)); \( x_{1:M} \) denotes the vertical concatenation of \( \{x_1, \ldots, x_M\} \); \( O_{N \times M} \) (resp. \( I_N \)) denotes the \( N \times M \) null (resp. identity) matrix; \( \Omega_N \) (resp. \( 1_N \)) denotes the null (resp. ones) vector of length \( N \); \( \nabla_a \{b(a)\} \subseteq \mathbb{C}^N \) and \( \mathcal{H}_a \{b(a)\} \subseteq \mathbb{C}^{N \times D} \) denote the gradient and Hessian of \( b(a) : a \in \mathbb{R}^N \rightarrow \mathbb{C} \), while \( J_{\mathcal{C}} \{e(x)\} \subseteq \mathbb{C}^{N \times P} \) is the Jacobian of \( e(x) : x \in \mathbb{R}^N \rightarrow \mathbb{C}^P \); \( E_{\mathcal{C}} \{\cdot\} \), \( \cdot^T \), \( \cdot^* \), \( \cdot^\dagger \), \( \cdot^{\text{tr}} \), \( \text{Re} \{\cdot\} \), \( \text{Im} \{\cdot\} \), \( \text{angle} \), \( \text{det} \), \( ||\cdot||_p \) and \( ||\cdot|| \) denote expectation, transpose, Hermitian, matrix trace, pseudo-inverse, real part, phase, Kronecker delta, Frobenius and \( L_2 \) norms operators, respectively; \( \text{diag}(a) \) is the diagonal matrix obtained from the vector \( a \); \( \text{vec}(M) \) stacks the first to the last columns of the matrix \( M \) one under another to form a vector; \( \Sigma_{m} \) is the covariance matrix of the complex-valued random vector \( x \); \( \mathcal{N}_{\mathbb{C}}(\mu, \Sigma) \) denotes a proper complex Gaussian pdf with mean vector \( \mu \) and covariance matrix \( \Sigma \); \( \mathcal{U}(a, b) \) denotes a uniform pdf with continuous support \( [a, b] \); finally \( \sim_{\text{~}} \) means “distributed as”.

3The number of scatterers \( M \) is assumed to be known [6].
as $\tau \triangleq \begin{bmatrix} \tau_1 & \cdots & \tau_M \end{bmatrix}^\top \in \mathbb{C}^M$. In Eq. (2) the Tx/Rx array matrix $G(x_{1:M}) \in \mathbb{C}^{N \times M}$ is defined as

$$G(x_{1:M}) \triangleq \begin{bmatrix} g(x_1) & g(x_2) & \cdots & g(x_M) \end{bmatrix},$$

(3)

where $g(x) \in \mathbb{C}^N$ denotes the Tx/Rx Green’s function vector as a function of the arbitrary location $x \in \mathbb{R}^p$, that is:

$$g(x) \triangleq \begin{bmatrix} G(r_1, x) & G(r_2, x) & \cdots & G(r_N, x) \end{bmatrix}^\top.$$  

(4)

It is worth noticing that the functional dependence of Eq. (4) is only due to $G(x', x)$, which denotes the relevant (scalar) background Green function [1]. Finally, in Eq. (2) the matrix $D(x_{1:M}, \tau) \in \mathbb{C}^{M \times M}$ for BA model [1] is defined as

$$D(x_{1:M}, \tau) \triangleq T(\tau) = \text{diag}(\tau),$$

(5)

while in the case of FL model we have [3]

$$D(x_{1:M}, \tau) \triangleq \left[ T^{-1}(\tau) - S(x_{1:M}) \right]^{-1},$$

(6)

with the $(m,n)$th element of $S(x_{1:M}) \in \mathbb{C}^{M \times M}$ defined as:

$$s_{m,n}(x_{1:M}) \triangleq \begin{cases} G(x_m, x_n) & m \neq n \\ 0 & m = n. \end{cases}$$

(7)

Both models will be considered in our analysis of TR-MUSIC.

A. TR-MUSIC Pseudospectrum

TR-MUSIC evaluates the spatial spectrum [1]

$$P(x, \bar{U}_n) \triangleq \left\| \bar{U}_n^\dagger g(x) \right\|^2 = \left\| g(x) \right\|^2 \bar{P}_n g(x),$$

(8)

where $\bar{U}_n \in \mathbb{C}^{N \times (N-M)}$ is the matrix of (noise subspace related) left singular vectors of $K_n$ (cf. Eq. (2)) and $\bar{P}_n \triangleq \bar{U}_n^\dagger \bar{U}_n$ (i.e. the corresponding projector). Eq. (8) equals zero when $x$ equals the true scatterers locations $\{x_k\}_{k=1}^M$ in the noise-free case and thus the $M$ largest local maxima of $P(x, \bar{U}_n)^{-1}$ are generally chosen as the estimates $\{\hat{x}_k\}_{k=1}^M$.

B. Preliminaries on SVD Perturbation

Here we give preliminaries on first-order SVD perturbation [11], [12]. Let us consider the perturbed matrix $\hat{A} = (A + N) \in \mathbb{C}^{R \times R}$, where $A$ and $N$ are the unperturbed (with rank $\delta < R$) data and the perturbing matrices, respectively. The SVD of $A = U \Sigma V^\dagger$ is expanded as (we define $\delta \triangleq (R-\delta)$):

$$A = \begin{pmatrix} U_s & U_n \end{pmatrix} \begin{pmatrix} \Sigma_s & \delta \times \delta \\ \delta \times \delta & \Sigma_n \end{pmatrix} \begin{pmatrix} V_n^\dagger \\ V_n^\dagger \end{pmatrix},$$

(9)

while the SVD of $\hat{A} = \bar{U} \Sigma \bar{V}^\dagger$ is expressed as:

$$\hat{A} = \begin{pmatrix} U_s & U_n \end{pmatrix} \begin{pmatrix} \Sigma_s & \delta \times \delta \\ \delta \times \delta & \Sigma_n \end{pmatrix} \begin{pmatrix} \bar{V}_n^\dagger \\ \bar{V}_n^\dagger \end{pmatrix}.$$  

(10)

We can then write:

$$\hat{U}_n = U_n + \Delta U_n,$$

(11)

where $\Delta U_s$ and $\Delta U_n$ are the perturbations in the estimated (left) signal and noise subspaces, respectively. At high signal-to-noise (SNR) ratio, a first-order perturbation will be accurate [12]. The expression for $\Delta U_n$, valid up to the first order, is:

$$\Delta U_n = U_s B = -U_s \Sigma_s^{-1} V_n^\dagger N^\dagger U_n.$$  

(12)

On the other hand $\Delta U_s = U_n C$, where $C = -B^\dagger$, thus giving ($P_n \triangleq U_n U_n^\dagger$):

$$\Delta U_s = U_n C = P_n N V_n \Sigma_s^{-1}.$$  

(13)

III. PERFORMANCE ANALYSIS

Here we generalize the classical steps used for MUSIC performance analysis in DOA estimation [12] to our model. It is known that, for true scatterers location, we have:

$$P(x_k, \bar{U}_n) = 0, \quad k \in \{1, \ldots, M\}$$

(14)

However, due to the perturbing matrix $W$, TR-MUSIC obtains an estimate $\hat{x}_k = (x_k + \Delta x_k)$. We first use the Newton-Raphson method to approximate $\Delta x_k$ as:

$$\Delta x_k \approx -\left( \mathcal{H}_x \{P(x, \bar{U}_n)\}^{-1} \nabla_x \{P(x, \bar{U}_n)\} \right)_{x=x_k}$$

(15)

For notational simplicity we denote $n(x, \bar{U}_n) \triangleq -\nabla_x \{P(x, \bar{U}_n)\}$ and $D(x, \bar{U}_n) \triangleq \mathcal{H}_x \{P(x, \bar{U}_n)\}$. Secondly, we express $n(x, \bar{U}_n)$ as:

$$n(x, \bar{U}_n) = -\nabla_x \{g(x)^\dagger P_n g(x)\}$$

$$= -2 \Re \left\{ J_k \{g(x)\}^\dagger P_n g(x) \right\}$$

(16)

(17)

The first-order perturbations of $n(x_k, \bar{U}_n)$ and $D(x_k, \bar{U}_n)$ with respect to $\Delta U_n$ are given by:

$$n(x_k, \bar{U}_n) \approx n(x_k, U_n) + \Delta n = n + \Delta n$$

(18)

$$D(x_k, \bar{U}_n) \approx D(x_k, U_n) + \Delta D = D + \Delta D$$

(19)

In the above equations $\Delta n$ and $\Delta D$ are both linear functions of $\Delta U_n$. Using the orthogonality property $U_n^\dagger g(x_k) = 0_{(N-M)}$, we have $n(x_k, U_n) = n = 0$. Hence, $\Delta x_k$ can be further approximated as follows:

$$\Delta x_k \approx (D(x_k, U_n) + \Delta D)^{-1} \Delta n$$

(20)

$$= (D + \Delta D)^{-1} \Delta n = (I + D^{-1} \Delta D)^{-1} D^{-1} \Delta n$$

(21)

$$= -\sum_{k=0}^{+\infty} \left( D^{-1} \Delta D \right)^k (D^{-1} \Delta n) \approx D^{-1} \Delta n$$

(22)

where in Eq. (22) we have kept only the first-order term of the Neumann series. The term $\Delta n$ can be evaluated from the closed form of $n(x_k, \bar{U}_n)$ (cf. Eq. (17)):

$$n(x_k, \bar{U}_n) = -2 \Re \left\{ J_k \{U_n^\dagger g(x_k)\} g(x_k) \right\} =$$

$$-2 \Re \left\{ J_k^\dagger U_n U_n^\dagger g(x_k) + J_k^\dagger \Delta U_n \{U_n^\dagger g(x_k)\} + J_k^\dagger U_n (\Delta U_n)^3 \{g(x_k)^\dagger\} g(x_k) \right\}$$

(23)

(24)

where we have used the short-hand notation $J_k \triangleq J_k \{g(x)\}$. It is apparent from Eq. (24) that both first and last terms are zero since $U_n^\dagger g(x_k) = 0_{(N-M)}$, while the second term is quadratic in $\Delta U_n$ and thus can be discarded in a first-order analysis. Thus, Eq. (24) is approximated by:

$$n(x_k, \bar{U}_n) \approx -2 \Re \left\{ J_k^\dagger U_n (\Delta U_n)^3 \{g(x_k)^\dagger\} g(x_k) \right\}$$

(25)
From direct comparison of Eqs. (18) and (25), we obtain:

$$\Delta n = -2 \Re\left\{ J_k U_n (\Delta U_n)^\dagger g(x_k) \right\}. \quad (26)$$

Differently, the explicit form of $D = D(x_k, U_n)$ is found as:

$$D = D(x_k, U_n) = \mathcal{H}_x \{ P(x, U_n) \}_{x=x_k} = \mathcal{H}_x \{ g(x) P_n g(x) \}_{x=x_k} = 2 \Re\left\{ J_k^\dagger P_n J_k \right\}$$

Therefore, in view of Eqs. (26) and (27), we obtain:

$$\Delta x_k \approx - \Re\{ J_k^\dagger P_n J_k \}^{-1} \Re\{ J_k^\dagger U_n (\Delta U_n)^\dagger g(x_k) \} = - \Re\{ J_k^\dagger P_n J_k \}^{-1} \Re\{ J_k^\dagger U_n (\Delta U_n)^\dagger g(x_k) \}$$

Now we substitute Eq. (12) in Eq. (28), thus leading to:

$$\Delta x_k \approx \Re\{ J_k^\dagger P_n J_k \}^{-1} \Re\{ J_k^\dagger P_n W K(x_{1:M}, \tau)^\dagger g(x_k) \}; \quad (29)$$

where we have exploited $K(x_{1:M}, \tau)^\dagger = V_s \Sigma_s^{-1} U_l^\dagger$. Aiming at notational simplicity, we define $\alpha_k \triangleq K(x_{1:M}, \tau)^\dagger g(x_k)$, $B_k \triangleq P_n J_k$ and $\Gamma_k \triangleq \Re\{ J_k^\dagger P_n J_k \}$, which allow us to rewrite Eq. (29) as:

$$\Delta x_k \approx \Gamma_k^{-1} \Re\{ B_k^\dagger W \alpha_k \}. \quad (30)$$

Since the first-order perturbation $\Delta x_k$ is linear in the noise matrix $W$, it follows immediately $\mathbb{E}\{\Delta x_k\} = 0_p$. Then the covariance matrix of $\Delta x_k$ equals the MSE matrix of $\hat{x}_k$ (i.e. the estimated position of $k$th target is unbiased at high SNR).

We can now evaluate $\Sigma_{\Delta x_k} = \mathbb{E}\{\Delta x_k \Delta x_k^\dagger\}$ explicitly. It is shown in the Appendix that the latter is given by:

$$\Sigma_{\Delta x_k} = \frac{1}{2} \sigma_w^2 \| \alpha_k \|^2 \Gamma_k^{-1} \Re\{ B_k^\dagger B_k \} (\Gamma_k^{-1})^\dagger \quad (31)$$

Further simplifications, exploiting analogous steps as in [12], are obtained as follows. First, we notice that $\Re\{ B_k^\dagger B_k \} = \Gamma_k$ (since $P_n = P_n^\dagger = P_n^2$) and $\Gamma_k = \Gamma_k^\dagger$. Secondly, we use the definition $\alpha_k = K(x_{1:M}, \tau)^\dagger g(x_k)$, which leads to a reduced expression for Eq. (31):

$$\Sigma_{\Delta x_k} = \frac{\sigma_w^2}{\omega} \| K(x_{1:M}, \tau)^\dagger g(x_k) \|^2 \Re\{ J_k^\dagger P_n J_k \}^{-1} \quad (32)$$

The above result is representative of TR-MUSIC ability of localizing only $M < N$ targets since, when $M \geq N$, $P_n$ is null and thus an infinite covariance is predicted (cf. Eq. (32)). Also, the term $\| K(x_{1:M}, \tau)^\dagger g(x_k) \|^2$ determines the MSE dependence of TR-MUSIC on the particular scattering model considered (indeed this is the sole difference in Eq. (32) between FL and BA models, since it can be shown that $P_n$ is identical in both cases).

IV. SIMULATION RESULTS

For our simulations we consider a 2-D propagation in a homogeneous background, that is $\mathcal{G}(x', x) = \mathcal{H}_0 (\kappa^0 \| x' - x \|)$ (we drop the constant term $\frac{1}{2}$), where $\mathcal{H}_0(\cdot)$ denotes the $n$th order Hankel function of the first kind. In this case the $i$th row of $J_x \{ g(x) \}$ (being equal to $\nabla_{x_i} \{ \mathcal{G}(r_i, x) \}$) is obtained as:

$$\nabla_{x_i} \{ \mathcal{G}(r_i, x) \} = \kappa^0 \mathcal{H}_1 (\kappa^0 \| r_i - x \|) \frac{(r_i - x)}{\| r_i - x \|}$$

where in Eq. (33) we exploited the recursions in Hankel functions derivatives [13]. Here we define \( SNR \triangleq \frac{\| K \|^2}{N^2 \sigma_w^2} \), and we consider a setup where $\lambda = 1$ (thus $\kappa^0 = 2\pi$) and a $\frac{\lambda}{2}$-spaced Tx/Rx array of $N = 11$ elements is employed, as shown in Figs. 1-3. Our examples consider $M = 2$ sources.

We first consider a setup with $x_1 \approx [-1 - 6]^T$ (see Fig. 1) and $\tau = [3 \ 4]^T$. Fig. 2 shows MSE$_k$ vs. SNR (dB) of: (i) TR-MUSIC simulated performance (i.e. $\text{Tr}[\mathbb{E}\{\| \hat{x}_k - x_k \|^2\}]$), based on $5 \cdot 10^3$ Monte Carlo runs; (ii) TR-MUSIC theoretical results (obtained via Eq. (32)) and (iii) CRLB-based evaluation$^4$. Results for both BA and FL models are shown. First, TR-MUSIC simulations approach theoretical curves as the SNR grows (with both models), thus confirming our findings. Further, we notice that (theoretical) MSE$_k$ of TR-MUSIC with BA model is significantly lower than that with FL scattering, thus showing an opposite trend with respect to the CRLB (other than a different asymptotic predicted performance). Such a result may seem counter-intuitive at first glance; however this is explained since TR-MUSIC is a sub-optimal estimator and thus it does not achieve the CRLB (as the MLE) at high SNR.

In order to investigate this effect, Fig. 4 reports a scatter plot of $(\gamma_1, \gamma_2)$, where $\gamma_k \triangleq \frac{\text{CRLB}(x_k)}{\text{MSE}(x_k)}$ denotes the MSE of $k$th target obtained through CRLB [3]. The experiment is obtained by taking 200 realizations of the following randomized setup: $r_{tk} \sim \mathcal{U}(0, \lambda), \angle r_{tk} \sim \mathcal{U}(0, 2\pi)$, $x_k = [-8 + 16 d_{x,k} \quad (-4 - 8 d_{y,k})]^T$, where $d_{x,k} \sim \mathcal{U}(0, \lambda)$ and $d_{y,k} \sim \mathcal{U}(0, \lambda)$ (see Fig. 3). We observe that MSE of TR-MUSIC is significantly higher than that predicted by CRLB, both for BA and FL models (indeed $\gamma_k$ is always $< 1$). Also, the couple $(\gamma_1, \gamma_2)$ is less spread in the BA case; this can be explained since TR-MUSIC relies only on the noise subspace of $K_n$ and thus non-linearity of FL model is not effectively exploited, as opposed to the case of MLE [3].

V. CONCLUSIONS

In this paper we provided a theoretical performance analysis for TR-MUSIC, based on first-order SVD perturbation framework. We obtained the MSE matrix of localization error in closed form for a co-located multistatic scenario. Theoretical performances were confirmed by simulations and were shown to predict MSE of TR-MUSIC more accurately than CRLB.

$^4$The closed form of CRLB for BA and FL models is omitted here for the sake of brevity and can be found in [3].
Figure 2. MSE$e_k \triangleq \text{Tr}[\mathbb{E}(\|\hat{x}_k - x_k\|^2)]$ vs. SNR (dB) for TR-MUSIC (simulated - dashed lines, theoretical - solid lines) and CRLB (dot-dashed lines) with both BA and FL models.

Figure 3. Scatterers distribution in an example with 200 Monte Carlo runs.

APPENDIX

Here we provide the derivation of Eq. (31). We start by exploiting Eq. (30) in the definition of $\Sigma \Delta x_k$:

$$\Sigma \Delta x_k = \Gamma_k^{-1} \mathbb{E}\{\Re\{B_k^H W \alpha_k\}\Re\{B_k^H W \alpha_k\}^\dagger\}(\Gamma_k^{-1})^t$$  \hspace{1cm} (34)

Eq. (34) requires the evaluation of the inner expectation term. After some manipulations, we get:

$$\Xi \triangleq \mathbb{E}\{\Re\{B_k^H W \alpha_k\}\Re\{B_k^H W \alpha_k\}^\dagger\} = \frac{1}{4} \left\{B_k^H \mathbb{E}(W \alpha_k \alpha_k^\dagger W^\dagger) B_k + B_k^H \mathbb{E}(W \alpha_k \alpha_k^\dagger W^\dagger) B_k^\dagger + B_k^H \mathbb{E}(W \alpha_k \alpha_k^\dagger W^\dagger) B_k^\dagger + B_k^H \mathbb{E}(W \alpha_k \alpha_k^\dagger W^\dagger) B_k^\dagger \right\}$$  \hspace{1cm} (35)

We evaluate the expectations in Eq. (35) as follows:

$$\mathbb{E}(W \alpha_k \alpha_k^\dagger W^\dagger) = \sum_{m=1}^N \sum_{n=1}^N \alpha_{k,m} \alpha_{k,n}^* \mathbb{E}\{w_m w_n^\dagger\}$$  \hspace{1cm} (36)

$$\mathbb{E}(W \alpha_k \alpha_k^\dagger W^\dagger) = \sum_{m=1}^N \sum_{n=1}^N \alpha_{k,m} \alpha_{k,n}^* \mathbb{E}\{w_m w_n^\dagger\}$$  \hspace{1cm} (37)

where we have denoted the $n$th column of $W$ with $w_n$. Since $\text{vec}(W) \sim N_{\mathbb{C}}(0_{N^2}, \sigma_w^2 I_{N^2})$, the properties (i) $\mathbb{E}\{w_m w_n^\dagger\} = \delta_{m,n} \sigma_w^2 I_N$ and (ii) $\mathbb{E}\{w_m w_n^\dagger\} = O_{N \times N}$ hold. The latter are exploited in Eqs. (36-37), in order to show:

$$\mathbb{E}(W \alpha_k \alpha_k^\dagger W^\dagger) = \|\alpha_k\|^2 \sigma_w^2 I_N;$$  \hspace{1cm} (38)

$$\mathbb{E}(W \alpha_k \alpha_k^\dagger W^\dagger) = O_{N \times N}$$  \hspace{1cm} (39)

R.h.s. of Eqs. (38-39) are then replaced into (35), thus giving $\Xi = \frac{\sigma_w^2}{4} \|\alpha_k\|^2 \Re\{B_k^H B_k\}$. Direct substitution in Eq. (34) gives Eq. (31).

REFERENCES


Figure 4. Scatter plot of $(\gamma_1,\gamma_2)$ for both BA and FL models.