On the Maximal Invariant Statistic for Adaptive 
Radar Detection in Partially-Homogeneous 
Disturbance with Persymmetric Covariance 

D. Ciuonzo, Senior Member, IEEE, D. Orlando, Senior Member, IEEE, and L. Pallotta, Member, IEEE

Abstract—This letter deals with the problem of adaptive signal detection in partially-homogeneous and persymmetric Gaussian disturbance within the framework of invariance theory. First, a suitable group of transformations leaving the problem invariant is introduced and the Maximal Invariant Statistic (MIS) is derived. Then, it is shown that the (two-step) Generalized-Likelihood Ratio test, Rao and Wald tests can be all expressed in terms of the MIS, thus proving that they all ensure a Constant False-Alarm Rate (CFAR).

Index Terms—Adaptive radar detection, CFAR, invariance theory, maximal invariants, partially-homogeneous disturbance, persymmetric disturbance.

I. INTRODUCTION

ADAPTIVE detection has attracted enormous interest in the last decades (see e.g. [1], [2] and references therein). Most design solutions rely on the Homogeneous Environment (HE), wherein a set of secondary data (ideally free of useful signal) is available, sharing the same spectral properties of the disturbance as in the cell under test (primary data) [3], [4]. Though the HE often leads to elegant closed-form solutions ensuring satisfactory performance [5]–[7], relevant scenarios are often non-homogeneous due to environmental factors and system considerations [8]–[10]. Frequently, a non-homogeneous scenario is depicted through the Partially-Homogeneous Environment (PHE, which generalizes the HE), i.e., both the test data and secondary data share the same covariance up to an unknown scaling factor¹. The Adaptive Normalized Matched Filter (ANMF) [12], [13] (or Adaptive Coherence Estimator (ACE) [14]) is the most common detector employed in PHE. In fact, it corresponds to the Generalized Likelihood Ratio Test (GLRT) for the aforementioned model, as shown in [15], and to a two-step GLRT (2S-GLRT) design procedure². More recently, literature has shown that the ANMF also corresponds to Rao and Wald tests [16]. A general framework for adaptive detection in the PHE was also recently proposed in [17].

Interestingly, other studies concerning detectors design in PHE appeared in the literature in the last years. These include works exploiting the peculiarity of the PHE, along with the assumption of a persymmetric covariance structure, as adopted in devising a plain GLRT in [18]. The latter structure introduces dependences among the unknowns of the disturbance and can be exploited to reduce the number of secondary data needed for adaptive processors. More recently, Rao and Wald tests were derived according to the same philosophy in [19]. Similarly, a persymmetric version of ACE (referred to as “Per-ACE”) was obtained in [20], as the result of a 2S-GLRT technique in a PHE. Finally, in [21] GLRT and 2S-GLRT were obtained for the case of distributed targets.

All these works differ from [22], where the problem has been handled through the principle of invariance [23], [24]. Note however that [22] is restricted to the HE assumption. Indeed such principle allows to focus at the design stage on decision rules enjoying some desirable (practical) features. The first step consists in identifying a suitable group of transformations which leaves unaltered: (i) the formal structure of the hypothesis testing problem, (ii) the data distribution family and (iii) the useful signal subspace. Of course, the group invariance requirement leads to a (lossy) data reduction, with the least compression represented by the Maximal Invariant Statistic (MIS), organizing the original data into equivalence classes. Hence, every invariant test can be expressed in terms of the MIS [24]. Consequently, the parameter space is usually compressed after reduction by invariance and the dependence on the original parameters set is mapped into the induced maximal invariant [24]. Referring to radar adaptive detection, the mentioned principle represents the workhorse for obtaining a statistic which is invariant with respect to (w.r.t.) the set of nuisance parameters, thus constituting the enabler for Constant False-Alarm Rate (CFAR) detectors.

The contributions of this letter are summarized as follows. Unlike [22], the principle of invariance is exploited to obtain the MIS and, hence, invariant architectures, assuming the PHE and persymmetric covariance. Specifically, following the lead of [22], the problem at hand is recast in canonical form, which

¹Indeed, while keeping a relative tractability of the model, this assumption provides increased robustness to power level fluctuations of the disturbance between the test cell and the set of the training data, which may manifest for instance due to variations in terrain and the use of guard cells [8], [10], [11].

²This is tantamount to devising the GLRT under known covariance of the disturbance and then making it adaptive via its substitution with the sample covariance matrix of secondary data.
facilitates the derivation of the MIS and allows to gain insights on the problem. Then, the group of transformations which leaves the problem invariant is identified and the corresponding explicit expression of the MIS is derived. Remarkably, closed-form expressions for the (2S-) GLRT, the Rao test, and the Wald test [25] are derived and shown to be all function of the data solely through the MIS, thus proving their CFAReNess\(^3\).

The rest of the letter is organized as follows: in Sec. II, we formulate the problem under investigation; in Sec. III, we obtain the MIS for the problem at hand and show invariance for the considered detectors; finally, in Sec. IV we provide some concluding remarks. Proofs are confined to the Appendix and to a supplemental material document.

II. PROBLEM FORMULATION

In this section, we describe the detection problem at hand and recall its canonical form representation [22]. We assume that a sensing system collects data from \(N > 1\) (spatial and/or temporal) channels. The returns from the cell under test, after pre-processing, are properly sampled and arranged in \(r \in \mathbb{C}^{N \times 1}\). We want to test whether \(r\) contains useful target echoes or not. Additionally, we assume that a set of secondary (signal-free) data, \(r_k \in \mathbb{C}^{N \times 1}\), \(k = 1, \ldots, K\) (with \(K \geq 2N\)), is available. In summary, the decision problem at hand can be formulated in terms of the following binary hypothesis test

\[
\begin{align*}
H_0 : & \quad \begin{cases}
\{r = n_0, r_k = n_{0k}, k = 1, \ldots, K, \} \\
\{r = \alpha s + n, r_k = n_{0k}, k = 1, \ldots, K, \}
\end{cases} \\
H_1 : & \quad \begin{cases}
\{r = n_1, z_2 = n_2, \} \\
z_{1k} = n_{1k}, z_{2k} = n_{2k}, \quad k = 1, \ldots, K,
\end{cases}
\end{align*}
\]  

(1)

where

- \(s \in \mathbb{C}^{N \times 1}\) is the nominal steering vector (\(||s|| = 1\)), exhibiting a persymmetric structure, that is, \(s = Js^*\) with \(J \in \mathbb{R}^{N \times N}\) a suitably defined permutation matrix [26];
- \(\alpha \in \mathbb{C}\) is an unknown deterministic factor accounting for both channel reflectivity and propagation effects;
- \(n_0 \sim \mathcal{CN}(0_N, M_0)\) and \(n_{0k} \sim \mathcal{CN}(0_N, \gamma M_0)\), \(k = 1, \ldots, K\), where the positive definite covariance matrix \(M_0 \in \{R \in \mathbb{H}^{N \times N} : R = JR^*J\}\) and the scaling factor \(\gamma \in \mathbb{R}^+\) are both unknown deterministic quantities (the latter assumption determines a PHE).

The model in Eq. (1) can be recast in the more advantageous canonical form, as shown in [22]. Indeed, without loss of generality, we can express the problem as:

\[
\begin{align*}
H_0 : & \quad \begin{cases}
z_1 = n_1, z_2 = n_2, \\
z_{1k} = n_{1k}, z_{2k} = n_{2k}, \quad k = 1, \ldots, K,
\end{cases} \\
H_1 : & \quad \begin{cases}
z_1 = \alpha_1 e_1 + n_1, z_2 = \alpha_2 e_1 + n_2, \\
z_{1k} = n_{1k}, z_{2k} = n_{2k}, \quad k = 1, \ldots, K,
\end{cases}
\end{align*}
\]  

(2)

where we have adopted the notation \(z_1 \triangleq V \Re\{\mathcal{H}r\} \in \mathbb{R}^{N \times 1}\), \(z_2 \triangleq V \Im\{\mathcal{H}r\} \in \mathbb{R}^{N \times 1}\), \(z_{1k} \triangleq V \Re\{\mathcal{H}r_k\} \in \mathbb{R}^{N \times 1}\) and \(z_{2k} \triangleq V \Im\{\mathcal{H}r_k\} \in \mathbb{R}^{N \times 1}\), \(k = 1, \ldots, K\), for the transformed primary and secondary data, respectively. We recall that the unitary matrix \(T \in \mathbb{C}^{N \times N}\) (whose definition is provided in [26]) and the orthogonal matrix \(V \in \mathbb{R}^{N \times N}\) (any orthogonal matrix whose first row is aligned to \(T s\)) are needed to obtain an equivalent real-valued representation of the persymmetric model and rotate the space into the canonical basis, respectively. Additionally, \(\alpha_1 \triangleq \Re\{\alpha\}\) and \(\alpha_2 \triangleq \Im\{\alpha\}\) denote the unknown deterministic coefficients accounting for the useful signal (collected in the vector \(\mathbf{\alpha} \triangleq [\alpha_1 \alpha_2]^T\)), whereas \(e_1 \triangleq [1 \ 0 \ \cdots \ 0]^T \in \mathbb{R}^{N \times 1}\) denotes the steering vector in canonical representation. With reference to the disturbance, we have employed the analogous definitions \(n_1 \triangleq V \Re\{T n_0\}, n_2 \triangleq V \Im\{T n_0\}, n_{1k} \triangleq V \Re\{T n_{0k}\}\) and \(n_{2k} \triangleq V \Im\{T n_{0k}\}, \) \(k = 1, \ldots, K\), with \(n_0 \sim \mathcal{N}_N(0, M)\) and \(n_{0k} \sim \mathcal{N}_N(0, \gamma M), i = 1, 2, k = 1, \ldots, K\), where \(M \triangleq (1/2)V T M_0 V^H\) represents the transformed (real-valued) covariance matrix of the primary data.

Before proceeding further, we collect all the secondary data in \(Z_s \triangleq [z_{11} \ \cdots \ z_{1K} \ z_{21} \ \cdots \ z_{2K}] \in \mathbb{R}^{N \times 2K}\) and give the following preliminary definitions:\(^4\)

\[
Z_p \triangleq [z_1 \ z_2] = [Z_{2p}]^T, \quad S \triangleq Z_s Z_s^H = [s_{11} \ s_{12}; s_{21} \ s_{22}],
\]  

(3)

where \(z_{1p} \in \mathbb{R}^{1 \times 2}\) (i.e., a row vector), \(Z_{2p} \in \mathbb{R}^{(N-1) \times 2}\), \(s_{11} \in \mathbb{R}\) (i.e., a scalar), \(s_{12} \in \mathbb{R}^{N \times (N-1)}\) (i.e., a row vector), \(s_{21} \in \mathbb{R}^{(N-1) \times 1}\) and \(s_{22} \in \mathbb{R}^{(N-1) \times (N-1)}\) respectively. Furthermore, it is easily shown that \(Z_p H_1 \sim \mathcal{N}_N(0, \gamma M),\) whereas \(Z_s \sim \mathcal{N}_N(0, N_0, J_2, M),\) where \(M \sim \mathcal{N}_N(0, N_0, J_2, M)\).

In this letter we will consider decision rules which declare \(H_1\) (resp. \(H_0\)) if \(\Phi(Z_s, S) \geq \eta\) (resp. \(\Phi(Z_s, S) < \eta\)), where \(\Phi() : \mathbb{R}^{N \times 2} \times \mathbb{R}^{N \times N} \to \mathbb{R}\) indicates the generic form of a decision function based on the sufficient statistic\(^5\) \((Z_p, S)\) and \(\eta\) denotes the threshold set to ensure a desired false-alarm probability \(P_{fa}\).

III. MAXIMAL INVARIANT STATISTIC

In what follows, we will search for functions of data sharing invariance w.r.t. those parameters (namely, the nuisance parameters \(M\) and \(\gamma\)) which are irrelevant for the specific

\(^3\)Notation - Lower-case (resp. Upper-case) bold letters denote vectors (resp. matrices), with \(a_{n, m}\) (resp. \(A_{n,m}\)) representing the \(n\)-th (resp. the \(n, m\)-th) element of the vector \(a\) (resp. matrix \(A\)); \(R^N, \mathcal{C}^N, \) and \(\mathcal{H}^{N \times N}\) (resp. \(\mathbb{S}^{N \times N}\)) are the sets of \(N\)-dimensional vectors of real numbers, of complex numbers, and of \(N \times N\) Hermitian (resp. symmetric) matrices, respectively, while \(\Re^N\) denotes the set of positive-valued real numbers; \(E\{\cdot\}, (\cdot)^T, (\cdot)^H, \|\cdot\|_2, \|\cdot\|_F, \mathcal{R}\{\cdot\}\) and \(\mathcal{S}\{\cdot\}\), denote expectation, transpose, Hermitian, matrix trace, Euclidean norm, real part, and imaginary part operators, respectively; \(0_{N \times M}\) (resp. \(I_N\)) denotes the \(N \times M\) null (resp. identity) matrix; \(0_N\) (resp. \(1_N\)) denotes the null (resp. ones) column vector of length \(N\); \(\det(\cdot)\) denotes the determinant of matrix \(A\); \(A \otimes B\) indicates the Kronecker product between matrices \(A\) and \(B\); the symbol \(\sim\) means "distributed as"; \(\alpha \sim \mathcal{CN}(\mu, \Sigma)\) denotes a complex (proper) Gaussian-distributed vector \(\alpha\) with mean vector \(\mu \in \mathbb{C}^{N \times 1}\) and covariance matrix \(\Sigma \in \mathbb{R}^{N \times N}\); \(\mathcal{C}^{N \times M}(A, B, C)\) denotes a complex (proper) Gaussian-distributed matrix \(X\) with mean \(A \in \mathbb{C}^{N \times M}\) and Cov[vec(X)] = \(B \otimes C\).

\(^4\)These definitions will be thoroughly exploited in the derivation of the MIS in Sec. III.

\(^5\)In fact, Fisher-Neyman factorization theorem ensures that the optimal decision from \((Z_p, S)\) is tantamount to deciding from raw data [27].
decision problem. To this end, we resort to the so-called principle of invariance [24], whose main idea consists in finding transformations that properly cluster data without altering: (i) the formal structure of the hypothesis testing problem given by \( \mathcal{H}_0 : \|\boldsymbol{\alpha}\| = 0, \mathcal{H}_1 : \|\boldsymbol{\alpha}\| > 0 \); (ii) the Gaussian assumption for the received data under each hypothesis; (iii) the real symmetric structure of the covariance matrix and the useful signal subspace. Therefore, next subsection is devoted to the definition of a suitable group which fulfills the above requirements.

### A. Desired invariance properties

First, without loss of generality, we will consider transformations acting directly on the sufficient statistic \( \{\mathbf{Z}_p, \mathbf{S}\} \). Then, we denote by \( \mathcal{G}_N \) the linear group of (real-valued) \( N \times N \) non-singular matrices having the peculiar structure

\[
G \triangleq \begin{bmatrix} g_{11} & g_{12} \\ 0 & G_{22} \end{bmatrix},
\]

where \( g_{11} \neq 0 \) and \( \det(G_{22}) \neq 0 \). Also, let \( \mathcal{O}_2 \) represent the group of \( 2 \times 2 \) orthogonal matrices (with generic element denoted with \( U \)) and consider the set \( \mathbb{R}^+ \) (with generic element denoted with \( \varphi \)), along with the composition operator “\( \circ \)”. Defined as

\[
(G_a, U_a, \varphi_a) \circ (G_b, U_b, \varphi_b) = (G_b G_a, U_a U_b, \varphi_a \varphi_b).
\]

The sets and the composition operator are here represented compactly as \( \mathcal{L} \triangleq (\mathcal{G}_N \times \mathcal{O}_2 \times \mathbb{R}^+, \circ) \), in a group\(^b\) form. The group \( \mathcal{L} \) has the fundamental property of leaving the hypothesis testing problem in Eq. (2) invariant under the action \( \ell(\cdot, \cdot) \), defined as follows:

\[
\ell(\mathbf{Z}_p, \mathbf{S}) = (G \mathbf{Z}_p U, \varphi G \mathbf{S} G^T) \quad \forall (G, U, \varphi) \in \mathcal{L}.
\]

The proof of the aforementioned statement is straightforward and is omitted due to lack of space. Such property implies that \( \mathcal{L} \) preserves the family of distributions (i.e., \( G \mathbf{Z}_p U \) and \( \varphi G \mathbf{S} G^T \) remain Gaussian- and Wishart-distributed as \( \mathbf{Z}_p \) and \( \mathbf{S} \), respectively), along with the structure of the considered hypothesis testing problem. Additionally, \( \mathcal{L} \) is chosen to include all the transformations of practical interest, as they allow claiming the CFAR property (w.r.t. \( \mathcal{M} \) and \( \gamma \)) as a byproduct of the invariance.

### B. Derivation of the MIS

In Sec. III-A, we have identified a group \( \mathcal{L} \) which leaves the problem under investigation unaltered. As a consequence, we are thus reasonably motivated to search for decision rules that are invariant under \( \mathcal{L} \). To this end, we invoke the principle of invariance because it allows to construct statistics (viz. the MISs) that organize data into distinguishable equivalence classes (named orbits). Then, every invariant test (w.r.t. \( \mathcal{L} \)) can be written as a function of the corresponding maximal invariant [23].

\[^b\]Indeed \( \mathcal{L} \) satisfies the following elementary axioms: (i) it is closed w.r.t. the operation “\( \circ \)”; (ii) it satisfies the associative property and (iii) there exist both the identity and the inverse elements.

The MIS (w.r.t. \( \mathcal{L} \)) satisfies both the properties:

\[
\begin{cases}
(a) \quad \mathbf{T}(\mathbf{Z}_p, \mathbf{S}) = \mathbf{T}(\ell(\mathbf{Z}_p, \mathbf{S})), \forall \ell \text{ action of } \mathcal{L}, \\
(b) \quad \mathbf{T}(\mathbf{Z}_p, \mathbf{S}) = \mathbf{T}(\mathbf{Z}_p, \mathbf{S}^\prime) \Rightarrow \exists \ell \text{ action of } \mathcal{L}: (\mathbf{Z}_p, \mathbf{S}) = \ell(\mathbf{Z}_p, \mathbf{S}^\prime).
\end{cases}
\]

Conditions (a) and (b) correspond to invariance and maximality properties, respectively. The explicit expression of the MIS for the problem at hand is provided in the following proposition.

**Proposition 1.** A MIS w.r.t. \( \mathcal{L} \) for the problem in Eq. (2) is given by the vector:

\[
t(\mathbf{Z}_p, \mathbf{S}) \triangleq [t_1, t_2, t_3]^T = [\lambda_1/\lambda_4, \lambda_2/\lambda_4, \lambda_3/\lambda_4]^T,
\]

where \( \lambda_i, i = 1, 2 \) and \( \lambda_j, j = 3, 4 \), are the eigenvalues of \( \mathbf{\Psi}_0 \triangleq \mathbf{Z}_p^T \mathbf{S}^{-1} \mathbf{Z}_p \) and \( \mathbf{\Psi}_1 \triangleq \mathbf{Z}_{2p}^T \mathbf{S}_{22}^{-1} \mathbf{Z}_{2p} \), respectively.

**Proof:** The proof is given in Appendix A.

Some important remarks are now in order. The MIS is given by a 3-D vector, where the third component \( t_3 \) represents an ancillary part, that is, its distribution does not depend on the hypothesis in force. Furthermore, exploiting [24, Thm. 6.2.1], every invariant statistic may be written as a function of Eq. (9). Therefore, it follows that every invariant test is CFAR.

Finally, we conclude the section with a discussion on the induced maximal invariant [24], representing the reduced set of unknown parameters on which the hypothesis testing in the invariant domain depends. To this end, we observe that the pdf of \( t(\mathbf{Z}_p, \mathbf{S}) \) does not depend on \( \gamma \), given the normalization by \( \lambda_4 \). Thus, the induced maximal invariant is the same as in the case of a HE [22] and corresponds to the Signal-to-Interference plus Noise Ratio (SINR) \( \|\alpha\|^2 e_1^T M^{-1} e_1 \). As a result, when the hypothesis \( \mathcal{H}_0 \) is in force, the SINR equals zero and thus the pdf of \( t(\mathbf{Z}_p, \mathbf{S}) \) does not depend on any unknown parameter. This implies that every function of the MIS satisfies the CFAR property.

### C. Detectors design vs. the MIS

In this subsection we are concerned with the design of detectors based on theoretically-founded criteria. Accordingly, we will concentrate on the derivation of the well-known GLRT (including its two-step version), Rao, and Wald tests [25].

Before proceeding, we first report the explicit expressions of the Maximum Likelihood (ML) estimates of the scale parameters under both hypotheses \( \gamma_i, i = 0, 1 \):

\[
\begin{align*}
\hat{\gamma}_i &\triangleq \frac{\beta_i - (K + 1 - N) \text{Tr}[\mathbf{\Psi}_i]}{2(2K + 2 - N) \det(\mathbf{\Psi}_i)}, \\
\beta_i &\triangleq \sqrt{\text{Tr}[\mathbf{\Psi}_i]^2(K + 1 - N)^2 + 4N(2K + 2 - N)\det(\mathbf{\Psi}_i)},
\end{align*}
\]

The definition in (10) will be exploited to provide compact expressions of the following detectors. Indeed, the GLR is expressed as [18]:

\[
t_{\text{GLR}} \triangleq \frac{\hat{\gamma}_0 \frac{N}{\hat{\gamma}_1} \det(\mathbf{I}_2 + \hat{\gamma}_0 \mathbf{\Psi}_0)}{\hat{\gamma}_1 \frac{N}{\hat{\gamma}_0} \det(\mathbf{I}_2 + \hat{\gamma}_1 \mathbf{\Psi}_1)}
\]
while the 2S-GLR (viz. Per-ACE) is [20]:

$$t_{2s-\text{glr}} \triangleq \frac{\text{Tr}[\Psi_0]}{\text{Tr}[\Psi_1]}.$$  

(13)

Differently, the Rao statistic is given by [19]:

$$t_{\text{rao}} \triangleq \frac{\hat{\gamma}_0 \text{Tr} \left( [\Psi_0 - \Psi_1] (I_2 + \hat{\gamma}_0 \Psi_0)^{-2} \right)}{1 - \hat{\gamma}_0 \text{Tr} \left( [\Psi_0 - \Psi_1] (I_2 + \hat{\gamma}_0 \Psi_0)^{-1} \right)}.$$  

(14)

Finally, the Wald statistic is [19]:

$$t_{\text{wald}} \triangleq \frac{\hat{\gamma}_1 \left( \text{Tr}[\Psi_0] - \text{Tr}[\Psi_1] \right)}{.}.$$  

(15)

We now show that the aforementioned statistics are all functions of the data solely through the MIS $t(Z_p, S)$ (viz. the corresponding tests ensure a CFAR). The detailed proof of this claim is provided as supplementary material for this letter. Indeed, it is first proved that $t_{\text{glr}}$ in (12) can be rewritten as:

$$t_{\text{glr}} = \left( t_1 + t_2 \right) / \left( 1 + t_3 \right).$$  

(16)

Differently, proving this property for Rao statistic is much more involved and it is based on showing that the terms $\hat{\gamma}_0 \text{Tr} \left( [\Psi_0 - \Psi_1] (I_2 + \hat{\gamma}_0 \Psi_0)^{-2} \right)$ and $\hat{\gamma}_0 \text{Tr} \left( [\Psi_0 - \Psi_1] (I_2 + \hat{\gamma}_0 \Psi_0)^{-1} \right)$ are both invariant. Finally, Wald statistic can be rewritten as:

$$t_{\text{wald}} = g_\gamma(t_3) \left( (t_1 + t_2)^2 - (1 + t_3) \right),$$  

(17)

which proves its invariance.

IV. CONCLUSIONS

In this letter we studied adaptive detection of a point-like target in the presence of PHE with persymmetric-structured covariance by resorting to statistical theory of invariance. After obtaining the group of transformations leaving the hypothesis testing problem invariant (thus enforcing at the design stage the CFAR property), a MIS was derived for the aforementioned group. It was found that the MIS for the problem at hand is a 3-D vector, with the latter component being an ancillary term. Subsequently, we focused on the derivation of (2S-) GLR, Rao and Wald statistics for the considered problem. Remarkably, all the aforementioned statistics were shown to be functions of the data solely through the MIS and, hence, to share the invariance property w.r.t. $L$. This implied, as a consequence, that they ensure a CFAR w.r.t. the unknown parameters of the disturbance.

V. ACKNOWLEDGEMENT

The authors would like to thank Prof. A. De Maio, at University of Naples “Federico II”, for suggesting the topic and for the interesting discussions towards the solution of the problem.

APPENDIX A

PROOF OF PROPOSITION 1

The proof is based on the key observation that the action $\ell(\cdot, \cdot)$ (cf. Eq. (6)) can be re-interpreted as the sequential application of the following sub-actions:

$$\ell_1(Z_p, S) = (G Z_p U, G S G^T) \forall (G, U) \in \mathcal{L}_1,$n\ell_2(Z_p, S) = (Z_p, \varphi S) \forall \varphi \in \mathcal{L}_2,$n$$

(19)

where $\mathcal{L}_1 \triangleq \{ G_N \times O_2, \text{“} \circ \text{”} \}$ and $\mathcal{L}_2 \triangleq \{ \mathbb{R}^*, \text{“} \times \text{”} \}$ (i.e., the composition operator for $\mathcal{L}_2$ simply corresponds to the product). Then, it is recognized that the MIS for the sub-action $\ell_1(\cdot, \cdot)$ has been already obtained in [22], as the former represents the relevant action enforcing desirable invariances in a homogeneous background (viz. $\gamma = 1$). Such statistic is 4-D and given by $t_{\text{HE}}(Z_p, S) \triangleq (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T$, where $\lambda_1 \geq \lambda_2$ are the two eigenvalues of $Z_p^T S^{-1} Z_p$ and $\lambda_3 \geq \lambda_4$ denote the two eigenvalues of $Z_p^T S_p^{-1} Z_p$, respectively. Now, define the action $\ell_2^\prime(\cdot)$ acting on the couple of positive-valued scalars $a_i$, collected in the vector $\alpha \triangleq [a_1, a_2, a_3, a_4]^T$ (with $a_i$, corresponding to $\lambda_i$) as:

$$t_2^\prime(\alpha) = \varphi - 1 \alpha \quad \forall \varphi \in \mathcal{L}_2.$$  

(20)

It is not difficult to show that a MIS for the elementary operation $\ell_2^\prime(\cdot)$ in Eq. (20) is given by $t_2(\alpha) \triangleq \frac{[a_1, a_2, a_3, a_4]^T}{t^\prime(\alpha)}$. This is clearly achieved by verifying that both invariance and maximality properties [24] hold for $t_2(\cdot)$. Indeed, invariance follows from $t_2(\varphi^{-1} \alpha) = \left[ \varphi^{-1} a_1 \varphi^{-1} a_2 \varphi^{-1} a_3 \varphi^{-1} a_4 \right]^T = t_2(\alpha)$, while maximality can be proved as follows. Suppose that $t_2(\alpha) = t_2(\bar{\alpha})$ holds, which implies:

$$\bar{a}_i = \left( a_i / a_4 \right) a_i, \quad i = 1, 2, 3.$$  

(21)

Then there exists a $\varphi \in \mathcal{L}_2$, equal to $\varphi = \frac{a_i}{a_4}$, which ensures $(\varphi^{-1} \alpha) = \bar{\alpha}$. This demonstrates that $t_2(\bar{\alpha})$ is a MIS for $\ell_2^\prime(\cdot)$. Additionally, we notice that

$$t_{\text{HE}}(Z_p, S) = t_{\text{HE}}(Z_p, S) \Rightarrow t_{\text{HE}}(Z_p, \varphi S) = \frac{1}{\varphi} t_{\text{HE}}(Z_p, S), \quad \forall \varphi \in \mathcal{L}_2,$n$$

(22)

since $t_{\text{HE}}(Z_p, \varphi S) = \frac{1}{\varphi} t_{\text{HE}}(Z_p, S)$ holds for the problem at hand. Therefore, exploiting [24, p. 217, Thm. 6.2.2], it follows that a MIS for the action $\ell(\cdot, \cdot)$ is given by the composite function $t(Z_p, S) \triangleq t_2(\text{HE}(Z_p, S)) = [\lambda_1 / \lambda_3 \lambda_2 / \lambda_4 \lambda_3 / \lambda_4]^T$. This concludes our proof.

REFERENCES


